

# A non-boost-invariant solution of relativistic hydrodynamics in 1+3 dimensions

Yoshitaka Hatta,<sup>1</sup> Bo-Wen Xiao,<sup>2</sup> and Di-Lun Yang<sup>3</sup>

<sup>1</sup>*Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan*

<sup>2</sup>*Key Laboratory of Quark and Lepton Physics (MOE) and Institute of Particle Physics, Central China Normal University, Wuhan 430079, China*

<sup>3</sup>*Theoretical Research Division, Nishina Center, RIKEN, Wako, Saitama 351-0198, Japan*

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We present a new solution of relativistic hydrodynamics in 1+3 dimensions which depends on both the transverse coordinate and rapidity. At early times the flow expands dominantly longitudinally in a non-boost-invariant manner, and at late times it expands nearly spherically. These two regimes are shown to be related by symmetry. The effect of viscosity is also discussed.

## I. INTRODUCTION

There are two well-known solutions of the relativistic hydrodynamic equation which are intended to describe the evolution of the dense matter created in heavy-ion collisions. In Landau's picture [1], the colliding nuclei come to a complete halt and deposit energy in a region extended in the longitudinal (beam) direction. Subsequently, this region expands one-dimensionally into the vacuum due to the large longitudinal gradient. The Khalatnikov-Landau solution [1, 2] of relativistic hydrodynamics in 1+1 dimensions offers a concrete realization of this idea. On the other hand, in Bjorken's picture [3], the highly Lorentz-contracted nuclei pass through each other leaving behind a dense partonic region which expands longitudinally in a boost-invariant (rapidity-independent) manner. This is described by the Bjorken solution [3] and is considered to be a plausible picture at very high energy.

In realistic collisions, boost invariance is violated, and what actually happens is somewhere in between perfect stopping and perfect transparency. To accommodate this, there have been a number of attempts to interpolate the two solutions [4–11]. However, all of these works essentially deal with 1+1-dimensional hydrodynamics implicitly assuming, rather unrealistically, that the colliding nuclei have infinite transverse extent. As already observed by Landau [1], the one-dimensional expansion is eventually superseded by a three-dimensional one when the time reaches of the order of the nuclear transverse size. Yet, obtaining full-fledged 1+3-dimensional analytical solutions is quite challenging because the hydrodynamic equations intimately couple the longitudinal and transverse dynamics.

A notable exception is the solution obtained by Gubser [12]. This generalizes the Bjorken solution by adding transverse expansion while retaining boost invariance. Yet, like the 1+1-dimensional solutions, Gubser's solution depends only on two variables in a cleverly chosen coordinate system. Introducing the dependence on a third variable (rapidity or the azimuthal angle) is difficult and so far has been done in the form of small perturbations [13, 14] (see, however, [15–17]).

In this paper, we analytically construct a non-boost-invariant 1+3-dimensional solution of relativistic hydrodynamics which essentially depends on three variables and has appealing features as a model of low energy heavy-ion collisions. Namely, the flow expands dominantly longitudinally in a non-boost-invariant manner at early times  $t \ll L$  where  $L$  is the size of the nucleus, and at late times  $t \gg L$  it expands nearly spherically. This will be demonstrated fully analytically. In the intermediate regime  $t \sim L$ , we have not found a closed analytic expression. We however develop a perturbation theory to approach this regime expanding around the early/late-time solutions. At the end, we discuss the effect of viscosity by approximately solving the Navier-Stokes equation.

## II. HYDRODYNAMICS IN $dS_2 \times H_2$

Our starting point is the relativistic hydrodynamic equation for an ideal fluid

$$u^\mu \partial_\mu \varepsilon + (\varepsilon + p) \nabla_\mu u^\mu = 0, \quad (\varepsilon + p) u^\nu \nabla_\nu u^\mu + \Delta^{\mu\nu} \partial_\nu p = 0, \quad (1)$$

where  $\varepsilon$  is the energy density,  $p$  is the pressure and  $u^\mu$  is the flow velocity normalized as  $u^\mu u_\mu = -1$ .  $\nabla_\mu$  is the covariant derivative and  $\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$  is the projection operator transverse to the flow. We

assume the relativistic (conformal) equation of state  $p = \frac{1}{3}\varepsilon \propto T^4$  with  $T$  being the temperature. For our purpose, it is more convenient to rewrite (1) in a different, but equivalent form

$$\nabla_\mu(\sigma u^\mu) = 0, \quad (2)$$

$$u^\mu(\nabla_\mu(Tu_\nu) - \nabla_\nu(Tu_\mu)) = 0, \quad (3)$$

where  $\sigma \propto T^3$  is the entropy density. In the presence of conserved charges, one should couple the above equations with the continuity equation

$$\nabla_\mu(nu^\mu) = 0, \quad (4)$$

where  $n$  is the charge density. However, in ideal hydrodynamics (4) is not an independent equation and trivially solved by  $n \propto T^3$ .

As demonstrated in [12, 18, 19], if the equation of state is relativistic  $\varepsilon = 3p$ , a powerful method to construct nontrivial solutions is available. Instead of working in Minkowski space with the metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2, \quad (5)$$

one can work in different coordinate systems which are related to Minkowski space via the Weyl transformation. Consider, then, the following coordinate transformation

$$\begin{aligned} ds^2 &= -dt^2 + dx_\perp^2 + x_\perp^2 d\phi^2 + dz^2 = -d\tau_\perp^2 + \tau_\perp^2 d\eta_\perp^2 + \tau_\perp^2 \sinh^2 \eta_\perp d\phi^2 + dz^2 \\ &= \tau_\perp^2 \left( \frac{-d\tau_\perp^2 + dz^2}{\tau_\perp^2} + d\eta_\perp^2 + \sinh^2 \eta_\perp d\phi^2 \right), \end{aligned} \quad (6)$$

where  $x_\perp = \sqrt{x^2 + y^2}$  and  $\phi$  is the azimuthal angle. The ‘transverse proper time’  $\tau_\perp$  and the ‘transverse rapidity’  $\eta_\perp$  are defined as

$$\tau_\perp = \sqrt{t^2 - x_\perp^2}, \quad \eta_\perp = \frac{1}{2} \ln \frac{t + x_\perp}{t - x_\perp}. \quad (7)$$

The coordinates  $(\tau_\perp, \eta_\perp)$  were previously introduced in [20]. This coordinate system covers only the region  $t > x_\perp$ . The case  $x_\perp > t$  will be treated later. In the second line of (6), we observe that the Weyl rescaled metric  $ds^2/\tau_\perp^2$  is that of  $dS_2 \times H_2$ , the product of the two-dimensional de Sitter space and the two-dimensional hyperbolic space. Further transforming to the so-called global coordinates of  $dS_2$  (see Fig. 1), we arrive at

$$d\hat{s}^2 \equiv \frac{ds^2}{\tau_\perp^2} = -d\rho_\perp^2 + \cosh^2 \rho_\perp d\Theta^2 + d\eta_\perp^2 + \sinh^2 \eta_\perp d\phi^2, \quad (8)$$

where

$$\sinh \rho_\perp = \frac{\tau_\perp^2 - L^2 - z^2}{2L\tau_\perp}, \quad \tan \Theta = \frac{2Lz}{L^2 + \tau_\perp^2 - z^2}. \quad (9)$$

$L$  is an arbitrary length parameter, and can be considered as the transverse size of the nucleus. We shall work in the coordinates (8) and solve the hydrodynamic equations in the form (2) and (3). Solutions  $\{\hat{u}^\mu, \hat{\varepsilon}\}$  are then transformed back to Minkowski space via the formulas

$$u_\mu = \tau_\perp \frac{\partial \hat{x}^\nu}{\partial x^\mu} \hat{u}_\nu, \quad \varepsilon = \frac{\hat{\varepsilon}}{\tau_\perp^4}. \quad (10)$$

As a slight generalization, the above transformation can be combined with the time translation. Instead of (7), we may define

$$\tau_\perp = \sqrt{(t - t_0)^2 - x_\perp^2}, \quad \eta_\perp = \frac{1}{2} \ln \frac{t - t_0 + x_\perp}{t - t_0 - x_\perp}, \quad (11)$$

where  $t_0$  is arbitrary. In the following we only show the results for  $t_0 = 0$ , but one can freely replace  $t$  with  $t - t_0$ .

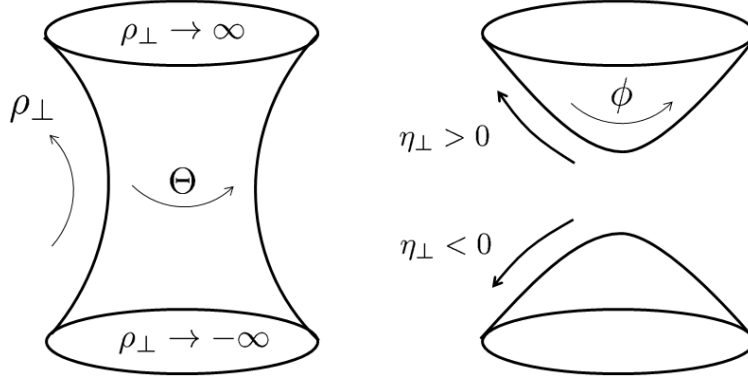


FIG. 1. The de Sitter space  $dS_2$  (left) and the hyperbolic space  $H_2$  (right).  $\rho_\perp$  plays the role of ‘time’ in the product space  $dS_2 \times H_2$ .

### A. An exact solution

Consider the comoving flow in  $dS_2 \times H_2$

$$(\hat{u}^{\rho_\perp}, \hat{u}^\Theta, \hat{u}^{\eta_\perp}, \hat{u}^\phi) = (1, 0, 0, 0). \quad (12)$$

With this velocity, (3) becomes trivial and (2) can be easily solved as

$$\hat{\varepsilon} \propto \left( \frac{1}{\cosh \rho_\perp} \right)^{4/3}. \quad (13)$$

The solution in Minkowski space is

$$(u_t, \vec{u}_\perp, u^z) = \frac{1}{\sqrt{(t^2 - x_\perp^2 - L^2 - z^2)^2 + 4L^2(t^2 - x_\perp^2)}} \times \left( \frac{t(t^2 - x_\perp^2 + L^2 + z^2)}{\sqrt{t^2 - x_\perp^2}}, \frac{\vec{x}_\perp(t^2 - x_\perp^2 + L^2 + z^2)}{\sqrt{t^2 - x_\perp^2}}, 2z\sqrt{t^2 - x_\perp^2} \right), \quad (14)$$

$$\varepsilon \propto \frac{1}{(t^2 - x_\perp^2)^{4/3}} \left( \frac{1}{4L^2(t^2 - x_\perp^2) + (t^2 - x_\perp^2 - L^2 - z^2)^2} \right)^{2/3}. \quad (15)$$

This is a new exact solution. It is analogous to Gubser flow by construction [12], but instead of expanding longitudinally in the  $z$ -direction, the fluid is expanding radially in the  $x_\perp$  direction at the speed of light. It is thus not an attractive model of heavy-ion collisions.

### III. THE NEW SOLUTION

#### A. Asymptotic solutions

We now consider a more general class of solutions of the form

$$(\hat{u}^{\rho_{\perp}}, \hat{u}^{\Theta}, \hat{u}^{\eta_{\perp}}, \hat{u}^{\phi}) = (\cosh \alpha, 0, \sinh \alpha, 0), \quad (16)$$

where  $\alpha = \alpha(\rho_{\perp}, \eta_{\perp})$  is the fluid rapidity in this space. Since the flow velocity (16) has only two components, the solution of (3) takes the form of potential flow in 1+1-dimensions

$$\hat{T}\hat{u}_{\rho_{\perp}} = -\hat{T} \cosh \alpha = \partial_{\rho_{\perp}} \Phi(\rho_{\perp}, \eta_{\perp}), \quad \hat{T}\hat{u}_{\eta_{\perp}} = \hat{T} \sinh \alpha = \partial_{\eta_{\perp}} \Phi(\rho_{\perp}, \eta_{\perp}). \quad (17)$$

In 1+1-dimensions, a standard way to proceed is to introduce the Khalatnikov potential  $\chi(\hat{T}, \alpha)$  [1, 2] as the Legendre transform of the potential  $\Phi(\rho_{\perp}, \eta_{\perp})$  and solve the resulting equation for  $\chi$ . However, the present problem does not fully reduce to a 1+1-dimensional one because the metric depends on  $\rho_{\perp}$  and  $\eta_{\perp}$ , and this makes the analysis in terms of  $\chi$  difficult. Instead, we develop a systematic perturbation theory to determine the function  $\alpha(\rho_{\perp}, \eta_{\perp})$  order by order.

For this purpose, we first observe that there are two simple solutions of (17)

$$\hat{T} = e^{\rho_{\perp}}, \quad \alpha = -\eta_{\perp}, \quad \Phi = -e^{\rho_{\perp}} \cosh \eta_{\perp}, \quad (18)$$

$$\hat{T} = e^{-\rho_{\perp}}, \quad \alpha = \eta_{\perp}, \quad \Phi = e^{-\rho_{\perp}} \cosh \eta_{\perp}. \quad (19)$$

Substituting these into (2), we find that (18) and (19) approximately solve (2) up to terms of order  $\mathcal{O}(e^{\pm 2\rho_{\perp}})$  in the infinite ‘past’  $\rho_{\perp} \rightarrow -\infty$  and infinite ‘future’  $\rho_{\perp} \rightarrow \infty$ , respectively. It is tempting to regard these solutions as the asymptotic behaviors of a single solution in the limits  $\rho_{\perp} \rightarrow \mp\infty$ . Such a solution would be an attractive model of heavy-ion collisions. Indeed, in Minkowski space, the limit  $\rho_{\perp} \rightarrow -\infty$  corresponds to early times  $\tau_{\perp} \ll L$  and (18) becomes

$$\begin{aligned} \varepsilon \propto \frac{\hat{T}^4}{\tau_{\perp}^4} &= \left( \frac{2L}{\sqrt{(L^2 + z^2 + x_{\perp}^2 - t^2)^2 + 4L^2(t^2 - x_{\perp}^2)} + L^2 + z^2 + x_{\perp}^2 - t^2} \right)^4 \\ &= \left( \frac{2L}{\sqrt{(L^2 + x_{\perp}^2 - \tau^2)^2 + 4L^2(\tau^2 \cosh^2 \eta - x_{\perp}^2)} + L^2 + x_{\perp}^2 - \tau^2} \right)^4, \end{aligned} \quad (20)$$

where in the second line we switched to the more familiar variables  $\tau = \sqrt{t^2 - z^2}$  and  $\eta = \frac{1}{2} \ln \frac{t+z}{t-z}$  often used in heavy-ion phenomenology. We see that the solution is not boost-invariant ( $\eta$ -dependent) and decays exponentially at large  $|\eta|$  for fixed  $\tau$ . Moreover,  $\varepsilon = \varepsilon(\tau, \eta, x_{\perp})$  depends on three variables in contrast to the Bjorken flow  $\varepsilon = \varepsilon(\tau)$  and the Gubser flow  $\varepsilon = \varepsilon(\tau, x_{\perp})$ . Note also that  $\varepsilon$  is finite in the limit  $\tau \rightarrow 0$ . As for the flow velocity, we find<sup>1</sup>

$$\begin{aligned} u_t &= -\frac{1}{t^2 - x_{\perp}^2} \left( \frac{t^2(t^2 - x_{\perp}^2 + L^2 + z^2)}{\sqrt{(t^2 - x_{\perp}^2 - L^2 - z^2)^2 + 4L^2(t^2 - x_{\perp}^2)}} - x_{\perp}^2 \right), \\ \vec{u}_{\perp} &= \frac{t\vec{x}_{\perp}}{t^2 - x_{\perp}^2} \left( \frac{t^2 - x_{\perp}^2 + L^2 + z^2}{\sqrt{(t^2 - x_{\perp}^2 - L^2 - z^2)^2 + 4L^2(t^2 - x_{\perp}^2)}} - 1 \right), \\ u_z &= \frac{2zt}{\sqrt{(t^2 - x_{\perp}^2 - L^2 - z^2)^2 + 4L^2(t^2 - x_{\perp}^2)}}. \end{aligned} \quad (21)$$

<sup>1</sup> In order to obtain this result we solved the defining equation (9) exactly for  $e^{\rho_{\perp}}$ . However, since (18) has been derived by neglecting terms of order  $e^{2\rho_{\perp}}$ , one could approximate  $\sinh \rho_{\perp} \approx -e^{-\rho_{\perp}}/2$ . If one does this, one finds a spherical flow at early times

$$\vec{u} \approx \frac{2t\vec{r}}{L^2 + r^2 - t^2},$$

where  $\vec{r} = (\vec{x}_{\perp}, z)$ . In order to resolve this ambiguity, one has to include the  $\mathcal{O}(e^{2\rho_{\perp}})$  corrections. This will be discussed below.

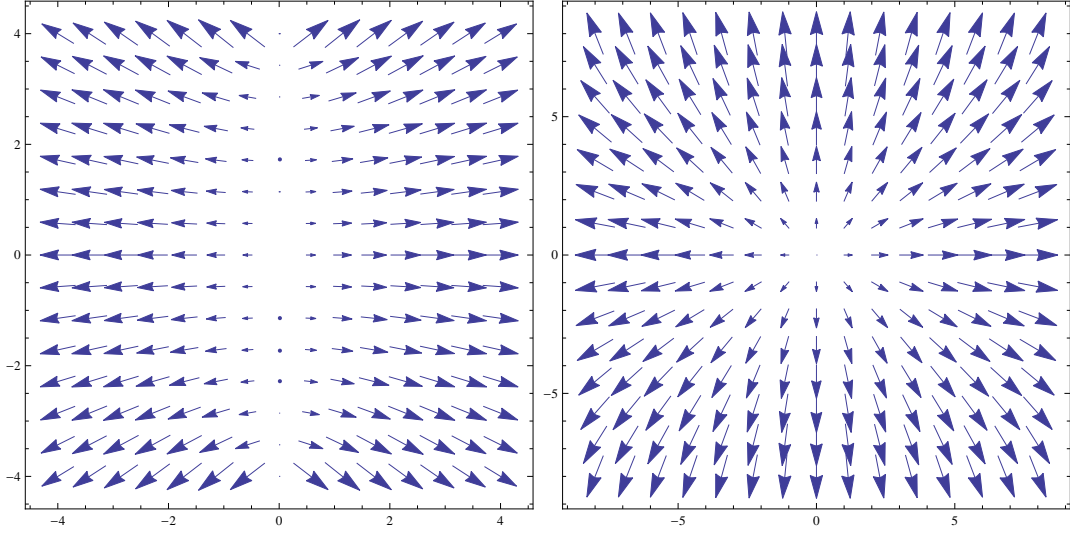


FIG. 2. The flow velocity  $(v_z, v_x)$  in the  $(z, x)$  plane for  $L = 4$ . Left: the early time solution (21) with  $t = 1$ . (As shown in Section III C, (21) is valid also for  $x_\perp > t$ .) Right: the late time solution (23) with  $t = 10$ .

Contrary to the appearance, there is no pole at  $t = x_\perp$  in  $u_t$  and  $u_\perp$ , and this suggests that the solution can be continued to  $x_\perp > t$  (see Section III C). The three-dimensional velocity  $(v_z, \vec{v}_\perp) \equiv (-u_z/u_t, -\vec{u}_\perp/u_t)$  is plotted in Fig.2(left). The flow is expanding dominantly longitudinally  $|v_z| \gg |v_\perp|$ , which is what one would expect in the early stages of heavy-ion collisions.

The other limit  $\rho_\perp \rightarrow \infty$  corresponds in Minkowski space to the late time regime  $\tau_\perp \gg L$  and (19) becomes

$$\varepsilon \propto \left( \frac{2L}{t^2 - x_\perp^2 - z^2 - L^2 + \sqrt{(t^2 - x_\perp^2 - z^2 - L^2)^2 + 4L^2(t^2 - x_\perp^2)}} \right)^4, \quad (22)$$

$$\begin{aligned} u_t &= -\frac{1}{t^2 - x_\perp^2} \left( \frac{t^2(t^2 - x_\perp^2 + L^2 + z^2)}{\sqrt{(t^2 - x_\perp^2 - L^2 - z^2)^2 + 4L^2(t^2 - x_\perp^2)}} + x_\perp^2 \right), \\ \vec{u}_\perp &= \frac{t\vec{x}_\perp}{t^2 - x_\perp^2} \left( \frac{t^2 - x_\perp^2 + L^2 + z^2}{\sqrt{(t^2 - x_\perp^2 - L^2 - z^2)^2 + 4L^2(t^2 - x_\perp^2)}} + 1 \right), \\ u_z &= \frac{2zt}{\sqrt{(t^2 - x_\perp^2 - L^2 - z^2)^2 + 4L^2(t^2 - x_\perp^2)}}. \end{aligned} \quad (23)$$

As shown in Fig. 2(right), the flow is almost spherical. In fact, this asymptotic late time regime is not reached in actual heavy-ion collisions because the system freezes out earlier, presumably when  $t \sim \mathcal{O}(L)$ . Yet, the transition from one-dimensional to three-dimensional expansions is expected on general grounds [1], and it is reassuring to see this analytically.

It is remarkable that, although the two solutions (18) and (19) are trivially related by the reflection symmetry  $\rho_\perp \rightarrow -\rho_\perp$  in  $dS_2 \times H_2$  (see Fig. 1), they appear quite distinct in Minkowski space. In fact, similar comments apply to Gubser's solution whose flow velocity reads, in our notation,

$$(v_z, v_\perp) = \left( \frac{z}{t}, \frac{x_\perp}{t} \frac{2\tau^2}{L^2 + \tau^2 + x_\perp^2} \right). \quad (24)$$

The flow is confined within the light cone  $|z| \leq t$  and there is a singularity at  $|z| = t$  where  $T$  diverges and  $|v_z|$  approaches unity irrespective of the value of  $x_\perp$ . Our solution has support also at  $|z| > t$  and has no singularity on the light-cone  $t = |z|$ .<sup>2</sup> It is thus more relevant to low energy collisions. Yet,

<sup>2</sup> The support property of  $\varepsilon$  and its physical interpretation is subject to the choice of  $t_0$  in (11). But in any case, clearly the light-cone  $t = |z|$  plays no role in our solution.

our solution is qualitatively different also from the Khalatnikov-Landau 1+1-dimensional solution. The rapidity dependence in the one-dimensional stage (20) is exponential at large  $|\eta|$

$$\sigma \sim e^{-3|\eta|}, \quad \varepsilon \sim e^{-4|\eta|}, \quad (25)$$

rather than the Gaussian-like distribution  $\sigma \sim e^{\sqrt{\#-\eta^2}}$  [1, 9]. Moreover, in the three-dimensional stage at late times the energy density decreases as (see (22))

$$\varepsilon \sim \frac{1}{t^8}, \quad (26)$$

in contrast to the  $\varepsilon \sim 1/t^4$  behavior estimated by Landau. By dealing with the fully 1+3-dimensional problem already in the early stage, we have arrived at a genuinely new type of solution.

## B. Perturbative expansion

Unfortunately, we have not found an exact analytical expression  $\alpha(\rho_\perp, \eta_\perp)$  which interpolates the limiting solutions (18), (19). We can however construct an approximate solution perturbatively in the form

$$\hat{\varepsilon} = e^{\pm 4\rho_\perp} \left( 1 + \sum_{k=1}^{\infty} a_k(\eta_\perp) e^{\pm 2k\rho_\perp} \right), \quad \alpha = \pm \left( -\eta_\perp + \sum_{k=1}^{\infty} b_k(\eta_\perp) e^{\pm 2k\rho_\perp} \right), \quad (\rho_\perp \rightarrow \mp\infty), \quad (27)$$

where we opt to solve in terms of  $\hat{\varepsilon}$  instead of  $\hat{T}$ . The coefficients  $a_k, b_k$  can be determined order by order by substituting (27) into (2) and (17), or more explicitly,

$$3(\cosh \alpha \partial_{\rho_\perp} \hat{\varepsilon} + \sinh \alpha \partial_{\eta_\perp} \hat{\varepsilon}) + 4\hat{\varepsilon}(\tanh \rho_\perp \cosh \alpha + \coth \eta_\perp \sinh \alpha + \sinh \alpha \partial_{\rho_\perp} \alpha + \cosh \alpha \partial_{\eta_\perp} \alpha) = 0, \quad (28)$$

$$\sinh \alpha \partial_{\rho_\perp} \hat{\varepsilon} + \cosh \alpha \partial_{\eta_\perp} \hat{\varepsilon} + 4\hat{\varepsilon}(\cosh \alpha \partial_{\rho_\perp} \alpha + \sinh \alpha \partial_{\eta_\perp} \alpha) = 0. \quad (29)$$

Let us make several general remarks about the structure of this perturbative expansion. (i) We take  $a_k$  and  $b_k$  to be common functions in the two regimes  $\rho_\perp > 0$  and  $\rho_\perp < 0$  (i.e.,  $a_k^+ = a_k^-$ ). The solution is then invariant under the global ‘time’ reversal  $\rho_\perp \rightarrow -\rho_\perp$ ,  $\hat{\varepsilon} \rightarrow \hat{\varepsilon}$ ,  $\alpha \rightarrow -\alpha$ , which is a property of the ideal hydrodynamic equation.<sup>3</sup> (ii) Eqs. (28) and (29) are invariant under the sign flip  $\eta_\perp \rightarrow -\eta_\perp$  provided  $\hat{\varepsilon}$  and  $\alpha$  are even and odd functions of  $\eta_\perp$ , respectively. This is indeed what comes out of the calculation. Geometrically, the sign flip  $\eta_\perp \rightarrow -\eta_\perp$  corresponding to jumping onto the other branch of the hyperbolic space  $H_2$  (see Fig. 1), though in practice  $\eta_\perp$  is positive by definition. (iii) From the analysis of the first few orders of the expansion (27), we noticed that  $a_k$  and  $b_k$  can be written as a linear combination of  $\cosh 2k'\eta_\perp$  and  $\sinh 2k'\eta_\perp$  with  $k' \leq k$ , respectively. This implies that in practice the expansion parameter is not  $e^{\pm 2\rho_\perp}$  but rather

$$e^{\pm 2\rho_\perp} \cosh^2 \eta_\perp \approx \begin{cases} t^2/L^2 & (t \ll L), \\ L^2/t^2 & (t \gg L). \end{cases} \quad (32)$$

Thus the expansion breaks down when  $t \sim L$ , and in this intermediate regime the solution can be constructed only numerically. Note that, if we shift the initial time as in (11), the expansion parameter is  $(t - t_0)^2/L^2$ . (iv) At each order of perturbation theory, one free parameter appears as an integration

<sup>3</sup> This in particular implies that  $\alpha$  has to vanish when  $\rho_\perp = 0$ , which is difficult to see from (27) (though of course (27) breaks down when  $\rho_\perp \rightarrow 0$ ). Instead, we may unify the two regimes  $\rho_\perp \gtrless 0$  and expand more symmetrically as

$$\hat{\varepsilon} = e^{\pm 4\rho_\perp} \left( 1 + \sum_{k=1}^{\infty} \frac{\tilde{a}_k(\eta_\perp)}{\cosh^{2k} \rho_\perp} \right), \quad \alpha = \eta_\perp \tanh \rho_\perp \left( 1 - \frac{1}{\eta_\perp} \sum_{k=1}^{\infty} \frac{\tilde{b}_k(\eta_\perp)}{\cosh^{2k} \rho_\perp} \right). \quad (30)$$

This is a very complicated reorganization of the series (27) and is not a consistent expansion if one truncates the sum to any fixed order. However, it makes the property  $\alpha(\rho_\perp = 0) = 0$  manifest. One can match the coefficients  $\tilde{a}_k$  and  $\tilde{b}_k$  with the ones in (27). For instance, at  $k = 1$  we have

$$\tilde{a}_1 = \frac{d_1 - 1}{6} - \frac{1}{6} (5d_1 + 1) \cosh 2\eta_\perp, \quad \tilde{b}_1 = -\frac{1}{2} \eta_\perp + \frac{d_1}{4} \sinh 2\eta_\perp, \quad (31)$$

where  $d_1$  is the same as in (33).

constant. Thus  $a_k$  and  $b_k$  contain  $k$  free parameters which in principle can be determined from the initial condition.<sup>4</sup>

Now let us show the  $k = 1$  solution which is relatively simple

$$a_1 = \frac{2}{3}(d_1 - 1) - \frac{2}{3}(5d_1 + 1) \cosh 2\eta_\perp, \quad b_1 = d_1 \sinh 2\eta_\perp, \quad (33)$$

where  $d_1$  is a free parameter. A particularly interesting choice is  $d_1 = 0$  in which case the correction to  $\alpha$  vanishes and the flow velocity (21) (and therefore Fig. 2(left)) is unmodified to this order. As  $d_1$  is increased from zero, the flow becomes rounder, and for negative  $d_1$  the transverse flow changes the directions. For phenomenological purposes one can simply set  $d_1$  to be zero. The  $k = 2$  solution is more involved

$$a_2 = \frac{2}{107} [37 + 33d_2 - 6d_1 + 84d_1^2 + (38 - 76d_2 + 150d_1 + 40d_1^2) \cosh 2\eta_\perp + 107d_2 \cosh 4\eta_\perp], \quad (34)$$

$$b_2 = \frac{\sinh 2\eta_\perp}{321} [17 + 180d_2 - (130 + 427d_1)d_1 + (20 - 468d_2 + (338 + 775d_1)d_1) \cosh 2\eta_\perp], \quad (35)$$

where  $d_2$  is the new free parameter which appears at this order. In the spirit that the correction to  $\alpha$  is made as small as possible, we may choose  $d_1 = 0$  and  $d_2 = \frac{5}{117}$  in which case

$$a_2 = \frac{2}{117} (42 + 38 \cosh 2\eta_\perp + 5 \cosh 4\eta_\perp), \quad b_2 = \frac{\sinh 2\eta_\perp}{13}. \quad (36)$$

Starting from  $k = 3$ , the equations become quite complicated and the coefficients of  $\cosh k'\eta_\perp$  and  $\sinh k'\eta_\perp$  ( $k' \leq k$ ) tend to become very large numbers. Rather than writing down the most general result which is not illuminating, here we only show the special solution obtained for  $d_1 = 0$ ,  $d_2 = \frac{5}{117}$ , and a particular value of the new parameter at  $k = 3$  which makes the coefficient of  $\sinh 6\eta_\perp$  in  $b_3$  vanish

$$a_3 = -\frac{2}{118989} (43573 + 42504 \cosh 2\eta_\perp + 5910 \cosh 4\eta_\perp + 355 \cosh 6\eta_\perp),$$

$$b_3 = \frac{1}{4407} (-282 \sinh 2\eta_\perp + 43 \sinh 4\eta_\perp). \quad (37)$$

In principle, we can continue this procedure to arbitrary higher orders. We however stop here because in the next section we shall see that already the  $k = 2$  terms are subleading compared with the viscous correction.

### C. Solution at $x_\perp > t$

By introducing the coordinates  $\tau_\perp, \eta_\perp$ , so far we have implicitly assumed that  $t > x_\perp$ . However, as we noted already, the flow profile (21) at early times has no singularity at  $t = x_\perp$ , and this suggests that the solution can be smoothly continued to  $x_\perp > t$ . To show that this is indeed the case, define for  $x_\perp > t$

$$\tilde{\tau} = \sqrt{x_\perp^2 - t^2}, \quad \tilde{\eta} = \frac{1}{2} \ln \frac{x_\perp + t}{x_\perp - t}. \quad (38)$$

We can then write

$$\begin{aligned} ds^2 &= -dt^2 + dx_\perp^2 + x_\perp^2 d\phi^2 + dz^2 = d\tilde{\tau}^2 - \tilde{\tau}^2 d\tilde{\eta}^2 + \tilde{\tau}^2 \cosh^2 \tilde{\eta} d\phi^2 + dz^2 \\ &= \tilde{\tau}^2 \left( -d\tilde{\eta}^2 + \cosh^2 \tilde{\eta} d\phi^2 + \frac{d\tilde{\tau}^2 + dz^2}{\tilde{\tau}^2} \right) \\ &= \tilde{\tau}^2 \left( -d\tilde{\eta}^2 + \cosh^2 \tilde{\eta} d\phi^2 + d\tilde{\rho}^2 + \sinh^2 \tilde{\rho} d\tilde{\Theta}^2 \right), \end{aligned} \quad (39)$$

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<sup>4</sup> The initial condition cannot be literally set at  $\rho_\perp = -\infty$  because  $\hat{T}$  vanishes there. The initial  $\rho_\perp$  has to be large but finite, and there are in principle infinitely many free parameters to specify the initial condition.

where

$$\cosh \tilde{\rho} = \frac{L^2 + \tilde{\tau}^2 + z^2}{2L\tilde{\tau}}, \quad \sinh \tilde{\rho} = \pm \frac{\sqrt{4z^2L^2 + (L^2 - \tilde{\tau}^2 - z^2)^2}}{2L\tilde{\tau}}, \quad \tan \tilde{\Theta} = \frac{L^2 - \tilde{\tau}^2 - z^2}{2Lz}. \quad (40)$$

The Weyl-transformed space  $ds^2/\tilde{\tau}^2$  is again  $dS_2 \times H_2$ . The  $\pm$  sign in (40) reflects the fact that the hyperbolic space  $H_2$  consists of two disconnected spaces. Noting that  $\tilde{\eta}$  is now the time-like variable, we look for solutions of the form

$$(\hat{u}^{\tilde{\eta}}, \hat{u}^{\phi}, \hat{u}^{\tilde{\rho}}, \hat{u}^{\theta}) = (\cosh \tilde{\alpha}, 0, \sinh \tilde{\alpha}, 0). \quad (41)$$

The (approximate) solution which matches with the previous solution (18) at  $t = x_{\perp}$  is obtained in the case  $\tilde{\rho} > 0$  and reads

$$\hat{T} = e^{-\tilde{\rho}}, \quad \tilde{\alpha} = \tilde{\eta}, \quad (\tilde{\rho} \rightarrow \infty). \quad (42)$$

One can again develop a perturbation theory around this solution. The solution in Minkowski space is identical to (20) and (21), and now we see that these results are valid both for  $t > x_{\perp}$  and  $t < x_{\perp}$ .

#### IV. VISCOUS CORRECTIONS

Finally in this section, we study the effect of viscosity. The shear viscosity  $\xi$  enters the nonequilibrium part of the energy momentum tensor

$$\delta \hat{T}^{\mu\nu} = -2\xi \hat{\sigma}^{\mu\nu}, \quad (43)$$

where  $\hat{\sigma}^{\mu\nu}$  is the shear tensor. In a conformal theory,  $\xi$  scales as

$$\xi \propto \hat{\varepsilon}^{3/4} \sim e^{\pm 3\rho_{\perp}}. \quad (44)$$

The hydrodynamic equation (1) is modified as

$$\hat{u}^{\mu} \partial_{\mu} \hat{\varepsilon} + \frac{4\hat{\varepsilon}}{3} \nabla_{\mu} \hat{u}^{\mu} = 2\xi \hat{\sigma}^{\mu\nu} \hat{\sigma}_{\mu\nu}, \quad (45)$$

$$4\hat{\varepsilon} \hat{u}^{\nu} \nabla_{\nu} \hat{u}^{\mu} + \Delta^{\mu\nu} \partial_{\nu} \hat{\varepsilon} = 6\Delta^{\mu}_{\nu} \nabla_{\lambda} (\xi \hat{\sigma}^{\lambda\nu}). \quad (46)$$

Let us compute  $\hat{\sigma}^{\mu\nu}$  for the  $k = 1$  solution (33)

$$\alpha = \pm (-\eta_{\perp} + d_1 \sinh 2\eta_{\perp} e^{\pm 2\rho_{\perp}}). \quad (47)$$

The nonvanishing components are found to be

$$\begin{aligned} \hat{\sigma}^{\rho_{\perp}\rho_{\perp}} &\approx \mp \frac{2}{3}(1-d_1) \cosh \eta_{\perp} \sinh^2 \eta_{\perp} e^{\pm 2\rho_{\perp}}, & \hat{\sigma}^{\rho_{\perp}\eta_{\perp}} &\approx \frac{2}{3}(1-d_1) \cosh^2 \eta_{\perp} \sinh \eta_{\perp} e^{\pm 2\rho_{\perp}}, \\ \hat{\sigma}^{\Theta\Theta} &\approx \pm \frac{4}{3}(1-d_1) \frac{\cosh \eta_{\perp}}{\cosh^2 \rho_{\perp}} e^{\pm 2\rho_{\perp}}, & \hat{\sigma}^{\phi\phi} &\approx \mp \frac{2}{3}(1-d_1) \frac{\cosh \eta_{\perp}}{\sinh^2 \eta_{\perp}} e^{\pm 2\rho_{\perp}}, \\ \hat{\sigma}^{\eta_{\perp}\eta_{\perp}} &\approx \mp \frac{2}{3}(1-d_1) \cosh^3 \eta_{\perp} e^{\pm 2\rho_{\perp}}. \end{aligned} \quad (48)$$

Due to a cancelation, the leading term is  $\hat{\sigma}^{\mu\nu} \sim \mathcal{O}(e^{\pm 2\rho_{\perp}})$ , and the two terms in (47) are equally important. In particular, the special value  $d_1 = 1$  makes  $\hat{\sigma}^{\mu\nu}$  vanish to this order. In fact, irrespective of the value of  $d_1$ , it is necessary to also retain the  $k = 2$  terms in  $\alpha$  which give  $\mathcal{O}(e^{\pm 4\rho_{\perp}})$  contributions to  $\hat{\sigma}^{\mu\nu}$ . To explain this, note that the right hand side of (45) is  $\mathcal{O}(e^{\pm 7\rho_{\perp}})$ , while that of (46) is naively  $\mathcal{O}(e^{\pm 5\rho_{\perp}})$ , but actually the coefficient of  $e^{\pm 5\rho_{\perp}}$  vanishes. The leading contribution is then  $\mathcal{O}(e^{\pm 7\rho_{\perp}})$ , and this comes from the  $\mathcal{O}(e^{\pm 4\rho_{\perp}})$  corrections to  $\hat{\sigma}^{\mu\nu}$  as well as the  $\mathcal{O}(e^{\pm 2\rho_{\perp}})$  corrections to the shear viscosity

$$\xi = \xi_0 \hat{\varepsilon}^{3/4} \approx \xi_0 e^{\pm 3\rho_{\perp}} \left( 1 + \frac{1}{2} (d_1 - 1 - (5d_1 + 1) \cosh 2\eta_{\perp}) e^{\pm 2\rho_{\perp}} \right), \quad (49)$$

where  $\xi_0$  is a constant. To the order of interest, (45) and (46) reduce to

$$\begin{aligned} 3(\cosh \alpha \partial_{\rho_{\perp}} \hat{\varepsilon} + \sinh \alpha \partial_{\eta_{\perp}} \hat{\varepsilon}) + 4\hat{\varepsilon}(\tanh \rho_{\perp} \cosh \alpha + \coth \eta_{\perp} \sinh \alpha + \sinh \alpha \partial_{\rho_{\perp}} \alpha + \cosh \alpha \partial_{\eta_{\perp}} \alpha) \\ = 16(1-d_1)^2 \xi_0 \cosh^2 \eta_{\perp} e^{\pm 7\rho_{\perp}}, \end{aligned} \quad (50)$$



$$\sinh \alpha \partial_{\rho_{\perp}} \hat{\varepsilon} + \cosh \alpha \partial_{\eta_{\perp}} \hat{\varepsilon} + 4\hat{\varepsilon}(\cosh \alpha \partial_{\rho_{\perp}} \alpha + \sinh \alpha \partial_{\eta_{\perp}} \alpha) = \pm C \xi_0 \sinh 2\eta_{\perp} e^{\pm 7\rho_{\perp}}, \quad (51)$$

where

$$C \equiv \frac{24}{107}(25 - 576d_2 + 88d_1 + 319d_1^2). \quad (52)$$

The solution to linear order in  $\xi_0$  is

$$\begin{aligned} \hat{\varepsilon} &\approx e^{\pm 4\rho_{\perp}} + \frac{2}{3}(d_1 - 1 - (5d_1 + 1) \cosh 2\eta_{\perp}) e^{\pm 6\rho_{\perp}} \pm \xi_0 A(\eta_{\perp}) e^{\pm 7\rho_{\perp}}, \\ \alpha &\approx \pm \left( -\eta_{\perp} + d_1 \sinh 2\eta_{\perp} e^{\pm 2\rho_{\perp}} \pm \xi_0 B(\eta_{\perp}) e^{\pm 3\rho_{\perp}} \right), \end{aligned} \quad (53)$$

where

$$\begin{aligned} A(\eta_{\perp}) &= \frac{1}{3} \left( \frac{4(1-d_1)^2}{3} - \frac{88D}{21} \pm \frac{5C}{14} \right) \cosh 3\eta_{\perp} + \left( \frac{4(1-d_1)^2}{3} + \frac{8D}{21} \mp \frac{3C}{14} \right) \cosh \eta_{\perp}, \\ B(\eta_{\perp}) &= D \sinh \eta_{\perp} \cosh 2\eta_{\perp} + \frac{1}{7} \left( D \pm \frac{3C}{4} \right) \sinh \eta_{\perp}, \end{aligned} \quad (54)$$

and  $D$  is an arbitrary constant. We see that the viscous effect brings in odd powers of  $e^{\pm\rho_{\perp}}$  in the series and is parametrically larger than the  $k=2$  corrections. Nevertheless,  $A, B$  are sensitive to the  $k=2$  flow velocity via the constant  $C$ . This is not inconsistent because we treat  $\xi_0$  as a small parameter and keep only the linear terms in  $\xi_0$ . The viscous effect may modify the  $k=2$  solution, but the backreaction of this onto the solution (56) via  $\hat{\sigma}^{\mu\nu}$  is of higher order in  $\xi_0$  and can be neglected. Note that the  $A, B$ -terms in (56) manifestly break the symmetry under  $\rho_{\perp} \rightarrow -\rho_{\perp}$ . This is because viscosity breaks the time-reversal symmetry of the hydrodynamic equations. As a consequence, the entropy is not conserved

$$\nabla_{\mu}(\sigma u^{\mu}) = \frac{16\xi_0}{3}(1-d_1)^2 \cosh^2 \eta_{\perp} e^{\pm 6\rho_{\perp}} + \mathcal{O}(e^{\pm 8\rho_{\perp}}). \quad (55)$$

To see the effect of viscosity quantitatively, as an illustration let us choose  $d_1 = 0$  and  $d_2 = \frac{5}{117}$  as was done in (36). We furthermore set  $D = 0$  to eliminate the  $\sinh 3\eta_{\perp}$  term in  $B(\eta_{\perp})$ . Then  $C = \frac{120}{1391}$  is negligibly small and the flow is still dominantly longitudinal at early times  $\alpha \approx -\eta_{\perp}$ . The energy density becomes

$$\varepsilon \approx \frac{1}{\tau_{\perp}^4} \left( e^{4\rho_{\perp}} - \frac{4}{3} \cosh^2 \eta_{\perp} e^{6\rho_{\perp}} + \xi_0 \frac{16}{9} \cosh^3 \eta_{\perp} e^{7\rho_{\perp}} \right) = T^4 \left( 1 - \frac{4}{3} t^2 T^2 + \frac{16\xi_0}{9} t^3 T^3 \right), \quad (56)$$

where  $T$  is as in (20). This is plotted in Fig. 3. The left and middle figures show that the first correction ( $k=1$ ) makes the flow more anisotropic. The viscous effects counteracts this change and tends to make the flow rounder, consistently with the general expectations. When  $D$  is increased from zero, the flow velocity  $u^{\mu}$  becomes relatively more isotropic. At the same time,  $A$  is reduced, and so is the viscous effect on  $\varepsilon$ .

In conclusion, we have presented a novel solution of 1+3-dimensional relativistic hydrodynamics which essentially depends on three variables  $\tau, \eta, x_{\perp}$  and qualitatively captures the salient features of the evolution of fireballs in heavy-ion collision. From our point of view, the initial one-dimensional expansion and the final three-dimensional expansion are geometrically related by the reflection symmetry  $\rho_{\perp} \rightarrow -\rho_{\perp}$  in the associated  $dS_2$  space. Our non-boost-invariant solution is essentially different from the boost-invariant Bjorken and Gubser solutions. Moreover, our fully 1+3-dimensional treatment has led to the exponential rapidity dependence (25), which makes this solution also distinct from the Khalatnikov-Landau solution.

Phenomenologically, our solution is more relevant to low energy heavy-ion collisions rather than high energy. The late time regime  $\rho_{\perp} \rightarrow \infty$  is presumably not reached in practice because the freezeout sets in earlier. Also, the value of  $t_0$  in (11) should be adjusted in order to mimic the initial matter distribution. We hope to return to these problems elsewhere.

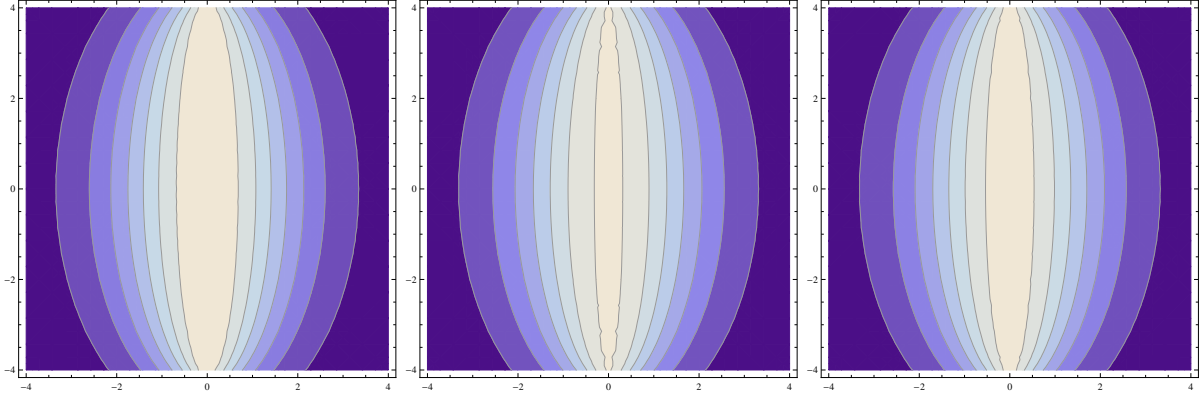


FIG. 3. The contour plot of the energy density  $\varepsilon$  in the  $(z, x)$  plane. Left: the first term of (56). Middle: the first and second terms of (56). Right: all terms of (56) included. We have set  $t = 1$  and  $L = 4$  as in Fig. 2. For an illustrative purpose, we used a somewhat large value  $\xi_0 = 1.5$ .

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